SEGMENTATION BASED ON MUMFORD-SHAH MODEL COMBINED WITH NARROW BAND*

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Abstract A segmentation model that combines the Mumford-Shah(M-S) model and narrow band scheme of level set was presented. The disadvantage of Mumford-Shah model is computationally time-consuming. In each step of its iteration, the data of whole image have to be renewed, which is unbearable for segmentation of large image or 3D image. Therefore, a fast segmentation model was introduce, which combines the M-S model and narrow band scheme by a new initialization method. The new initialization method is based on fast marching method, and the computing time decreases to $O(N)$. In each step of iteration, the new segmentation model only deals with the data in a narrow band instead of the whole image. The experiments show that the two models can obtain almost the same segmentation result, but the computing time of new narrow band M-S model is much less than that of M-S model.

Key words Mumford-Shah model, level set, narrow band scheme.

基于窄带 Mumford-Shah 模型的图像分割方法

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摘要 在结合 Mumford-Shah 模型和水平集方法中的窄带解法优点的基础上,提出了一种新的图像分割模型. Mumford-Shah模型虽然具有良好的图像分割结果,但是其每次迭代过程都需要对所有图像数据进行计算,因而很费时,导致这种方法不适用于大的图像数据,特别是三维图像的分割.本文通过一种新的初始化方法把 Mumford-Shah 模型和水平集中的窄带解法结合在一起.这种新的初始化方法是通过在特定条件下简化快速行进法得到的.通过去除快速步进法中费时的排序过程,使得初始化的计算时间只有 $O(N)$.窄带 Mumford-Shad 模型把分割计算限制在窄带范围内,避免了大量的计算,但取得了与原始的 Mumford-Shah 模型相同的分割效果.实验结果表明基于快速步进法的初始化方法是可行的,而窄带 M-S 分割模型一次迭代计算的时间比原 M-S 模型减少许多.

关键词 Mumford-Shah 模型,水平集方法,窄带法.

Introduction

The basic idea of active contour models or snakes is to evolve a curve subjecting to certain constraints and detect objects in that image. The classic snakes and active models[1,5,6,7,8,10] use an edge-detector according to the local gradient of the image, and stop the curve evolution on the boundary of the desired object. But these models only detect objects with sharp gradient edges. In case of the edges of low gradients and nonzero stopping function, the curve may pass through the boundary[3,4].

In order to overcome this disadvantage, the simplified Mumford-Shah(M-S) model was proposed by Chan and Vese[12]. The simplified M-S model can detect contour with or without gradient, for instance, at very smooth boundaries or even with discontinuous boundaries[11,13]. Although the M-S model is possible to obtain satisfactory segmentation results, the data of whole image have to be computed in each step of iteration and the computing time is unbearable, especially for big or three-dimensional image.

In this paper, we introduce a fast segmentation model, which combines the M-S model and narrow

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band scheme using a new initialization algorithm. The new initializtation algorithm modifie the fast marching algorithm by withdrawing the sorting step, and calculates the data in a narrow band. To accomplish the initialization, it only runs in $O(N)$ time where N is the total number of the pixels in the image.

1 Level set method and M-S model

1.1 Level set method

The level set method, proposed by Osher and Sethian[2], has the distinguished property of dealing with topology change.

Let $C(\mathrm{x},t)$ be a smooth, closed curve in Euclidean plane moving with time along its nomal vector field with speed F, x is the coordinate vector. Let $\Omega(\mathrm{x},t)$ be the region closed by $C(\mathrm{x},t)$. Define $\Phi(\mathrm{x},t)$ as the level set function associated with $\Omega(\mathrm{x},t)$.

$$\begin{cases}\Phi(\mathrm{x},t)>0 \text{ in } & \Omega(t)\\ \Phi(\mathrm{x},t)=0 \text{ on } & C(t)\\ \Phi(\mathrm{x},t)<0 \text{ in } & \Omega^c(t)\end{cases} \tag{1}$$

where $\Omega^c(\mathrm{x},t)$ is the complementary set of $\Omega(\mathrm{x},t)$. In equation (1), we can see that $C(\mathrm{x},t)$ is the zero level set of $\Phi(\mathrm{x},t)$. The evolving curve C is always zero in the level set function. So the evolving function of the curve is

$$\Phi(C(x,t),t)=0. \tag{2}$$

Differentiating the equation (2) with respect to t, leading to a time-dependent partial differntial equation.

$$\Phi_t+\frac{\mathrm{d}C(\mathrm{x},t)}{\mathrm{d}t}.\nabla\Phi=0\Leftrightarrow\Phi_t+F|\nabla\Phi|=0 \tag{3}$$

with a given value of $\Phi(C(\mathrm{x},t)=0)$. It is a kind of Hamilton-Jacobi equation.

In the equation (3), the position of the curve, which is also the position of the zero level set, can be traced in each iteration step, no matter how the topology of the curve changed. So, the level set method has the property of topology adaptability.

1.2 M-S model

The M-S model uses the global information of the image as the stopping criterion to segment the image.

Define the closed curve C in the image plane, which separates the plane Ω into two parts, outside and inside the curve. The method is to minimize an energy function, defined by

$$\begin{aligned}F(c_1,c_2,C) = {} & \mu\cdot\mathrm{Length}(C)+\\ & v\cdot\mathrm{Area}(\mathrm{inside}(C))+\\ & \lambda_1\int_{\mathrm{inside}}|u_0(x,y)-c_1|^2\mathrm{d}x\mathrm{d}y+\\ & \lambda_2\int_{\mathrm{outside}}|u_0(x,y)-c_2|^2\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{4}$$

where $\mu\geq0, v\geq0, \lambda_1, \lambda_2>0$ are fixed parameters, and c_1, c_2 are the average densities of regions inside and outside the curve, separately.

According to the level set formula, the unknown variable C is represented by the zero level set $\Phi=0$. Using the Heaviside function H

$$H(z)=\begin{cases}1, & if\ z\geq0\\ 0 & if\ z<0,\end{cases} \tag{5}$$

and the one-dimensional Dirac measure δ_0

$$\delta_0(z)=\frac{\mathrm{d}}{\mathrm{d}z}H(z).$$

The first and second terms in equation (4) can be expressed as follows

$$\begin{aligned}\mathrm{Length}(\Phi=0) &= \int_\Omega|\nabla H(\Phi(x,y))|\mathrm{d}x\mathrm{d}y\\ &= \int_\Omega\delta\Phi(x,y)|\nabla\Phi(x,y)|\mathrm{d}x\mathrm{d}y\\ \mathrm{Aera}(\Phi\geq0) &= \int_\Omega H(\Phi(x,y))\mathrm{d}x\mathrm{d}y\end{aligned} \tag{6}$$

and the energy function can be written as

$$\begin{aligned}F(\Phi,c_1,c_2) = {} & \mu\int_\Omega\delta(\Phi)|\nabla\Phi|\mathrm{d}x\mathrm{d}y+\\ & v\int_\Omega H(\Phi)\mathrm{d}x\mathrm{d}y+\\ & \lambda_1\int_\Omega|I-c_1|^2H(\Phi)\mathrm{d}x\mathrm{d}y+\\ & \lambda_2\int_\Omega|I-c_2|^2(1-H(\Phi))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{7}$$

In the equation (7), I is the gray level of the image pixel.

The above equation (7) is solved by Euler-Lagrangian equation.

$$\begin{cases} c_1(\Phi) = \dfrac{\int_\Omega I(x,y)H(\Phi)dxdy}{\int_\Omega H(\Phi)dxdy}, \\ c_2(\Phi) = \dfrac{\int_\Omega I(x,y)(1-H(\Phi))dxdy}{\int_\Omega (1-H(\Phi))dxdy}, \\ \dfrac{\partial \Phi}{\partial t} = \delta(\Phi)[\mu \nabla \cdot \dfrac{\nabla \Phi}{|\nabla \Phi|} - v - \\ \quad \lambda_1(I(x,y)-c_1)^2 + \lambda_2(I(x,y)-c_2)^2], \\ \Phi(0,x,y) = \Phi_0(x,y). \end{cases} \tag{8}$$

$c_1(\Phi)$, $c_2(\Phi)$ are the average gray level. And $\Phi(0, x, y)$ is the initial condition.

Then we get the iteration formulation

$$\frac{\Phi_{i,j}^{n+1} - \Phi_{i,j}^n}{\Delta t} = \delta_h \begin{bmatrix} \dfrac{\mu}{h^2}\Delta_-^x \left(\dfrac{\Delta_+^x \Phi_{i,j}^{n+1}}{\sqrt{(\Delta_+^x \Phi_{i,j}^n)^2/h^2 + (\Phi_{i,j+1}^n - \Phi_{i,j-1}^n)^2/h^2}}\right) \\ + \dfrac{\mu}{h^2}\Delta_-^y \left(\dfrac{\Delta_+^y \Phi_{i,j}^{n+1}}{\sqrt{(\Delta_+^y \Phi_{i,j}^n)^2/h^2 + (\Phi_{i,j+1}^n - \Phi_{i,j-1}^n)^2/h^2}}\right) \\ - v - \lambda_1(I_{i,j} - c_1(\Phi^n))^2 + \lambda_2(I_{i,j} - c_2(\Phi^n))^2 \end{bmatrix} \tag{9}$$

$\Phi_{i,j}^n$ is the value of level set function at the grid point (i,j) in the n iteration. $\Phi_{i,j}^{n+1}$ is the value in $n+1$ iteration. Δ_- and Δ_+ mean the backward and forward difference operators as follows.

$$\Delta_-^x \Phi_{i,j} = \Phi_{i,j} - \Phi_{i-1,j},$$
$$\Delta_+^x \Phi_{i,j} = \Phi_{i+1,j} - \Phi_{i,j},$$
$$\Delta_-^y \Phi_{i,j} = \Phi_{i,j} - \Phi_{i,j-1},$$
$$\Delta_+^y \Phi_{i,j} = \Phi_{i,j+1} - \Phi_{i,j}.$$

The M-S segmentation model overcomes the shortage of the classical snakes model by using the global information to make the curve stop at the edge of the object.

2 New initialization algorithm based on the fast marching algorithm

In this section, we introduce an initialization method based on the fast marching algorithm. Using the new initialization method, we can combine the M-S model and narrow band scheme of level set to get a fast segmentation model.

2.1 Fast marching method

The fast marching method is a numerical technique for solving the Eikonal equation[9,14].

Let $T(x,y,z)$ be the time at which the surface crossed a given point(x,y,z). The function $T(x,y,z)$ then satisfies the equation

$$|\nabla T| F = 1. \tag{10}$$

Using "entropy" condition[2,15], the following equation is obtained.

$$[\max(D_{i,j}^{-x}T,0)^2 + \min(D_{i,j}^{+x}T,0)^2 + \max(D_{i,j}^{-y}T,0)^2 + \min(D_{i,j}^{+y}T,0)^2]^{\frac{1}{2}} = \frac{1}{F_{i,j}}, \tag{11}$$

where D^- and D^+ are backward and forward difference operators as follows.

$$D_{i,j}^{-x}T = T_{i,j} - T_{i-1,j},$$
$$D_{i,j}^{+x}T = T_{i+1,j} - T_{i,j},$$
$$D_{i,j}^{-y}T = T_{i,j} - T_{i,j-1},$$
$$D_{i,j}^{+y}T = T_{i,j+1} - T_{i,j}.$$

The difference structure of equation(11) means that information propagates from small values of T to the larger values, which can be called and upwind fashion. The upwind fashion is determined by the following fact. The velocity F is derived from the image property, which is different from one point to another. When two points have the same geometrical distance to the curve point, but their speeds are not equal, consequently the reaching time of each point is not equal. The upwind fashion needs a sorting step to make sure that the points accepted previously have smaller reaching time than latter points. And the sorting step increases the marching time to $O(N \log(N))$.

If the speed term F is a constant, i.e. $F = c$, then starting from the initial front, the points will have same reaching time if their distances from the initial front are the same. So the sorting step can be omitted, which can save $O(\log(N))$ time and reduces the marching time to $O(N)$. In the initialization, the velocity is independent of the image property. So it is reasonable to have $F = c$.

2.2 A fast initialization method

The initialization algorithm classifies the points on the plane into three states:

(1) *Alive* means the point has been computed;

(2) *FarAway* means the point has not been computed;

(3) *Trial* means the point is in the waiting array.

Let the speed term $F = 1$, then the distance is numerically equal to the reaching time. So the value of reaching time can be substituted for level set distance.

The first step is to take the points whose distance is less than or equal to 1 into *Trial* region.

(1) Store the coordinates of all the points that are tagged as "CURVE"in an array CA.

(2) Access the array from head to tail in order. Let p_i be the accessing point. If p_i's distance is equal to zero, get the neighbor points $q_j, j \in (1,2,3,4)$ around the p_i.

(3) If q_j is not tagged as "CURVE", calculate the reaching time of q_j using the equation (11), insert q_j at the tail of another array NA, tag q_j as "NEIGHBOR". At the same time, record p_i as the nearest point on the curve of q_j.

(4) If p_i's distance is not equal to zero, calculate the distance of p_i and insert p_i into NA. Then find the corresponding point p_j of p_i on the other side of curve, calculate the distance of p_j and insert p_j at the tail of NA, tag p_j as "NEIGHBOR". And recording the nearest point of p_i as the nearest point of p_j.

(5) Loop the above procedure until all the points tagged as "CURVE"were accessed. And Get the new array NA.

The second step is using the array NA, get the distance of all the points in the image plane.

(1) Get the first point p_1 in NA, record it and delete it from NA. So every time we access the first element in NA.

(2) Obtain all the neighbor points $q_j, j \in (1,2,3,4)$ of p_1.

(3) If q_j has been accessed, omit it.

(4) If q_j has not been accessed, calculate its reaching time according to its neighbor points which has been accessed, insert q_j at the tail of the array NA, tag q_j as "NEIGHBOR". At the same time, record the nearest point of p_1 as the nearest point of q_j.

(5) Loop until all the image points are accessed.

3 Mumford-Shah model combined with narrow band

If segmenting objects in the whole image plane, it may be a time-consuming process. Fortunately, the level set method has a fast scheme named narrow band scheme. But the scheme can be only used after proper initialization. If the initialization uses the sweep algorithm proposed in [16], the narrow band scheme cannot be used correctly. Using the above initialization, we can combine the narrow band method of level set and M-S model to get a faster algorithm for image segmentation.

The narrow band is a neighbor around the curve with radius δ (Fig. 1). We only initialize when the curve evolves inside the narrow band. To form the narrow band, a threshold δ is given to stop the initialization. When the distance is bigger than δ, the point is not taken into the array NA. As the evolution curve reaches the edge of the narrow band, distance of all the points is bigger than δ and the array NA is empty. So the initialization stops automatically.

Then from the equation (9), we get the velocity of all the points in the narrow band. In the narrow band range, the M-S model is used to update the level set distance at each point, and evolve the initial toward the desired boundary. The whole procedure is as follows.

(1) Initialize Φ^n by Φ_0, $n=0$. Set a narrow band radius to get the narrow band region.

(2) Compute $c_1(\Phi^n)$ and $c_2(\Phi^n)$by equation (8)

(3) Extend the velocity on the current curve to all the points in the narrow band.

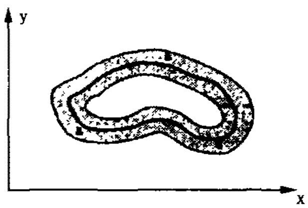

Fig. 1 A narrow band of width δ around the zero level set
图1 半径为 δ 的窄带

(4) Solve the PDE in Φ from equation (9), obtain Φ^{n+1}.

(5) Check whether the solution is stationary. If not, $n = n + 1$ and repeat; else, stop the loop.

4 Experiment results

Several experiments are given to show the segmentation results by proposed method. And the computational time comparison between the two models is given in Table 1.

In this experiment, the parameters are as follows: $\mu = 1, v = 0, \lambda_1 = \lambda_2 = 1, h = 1$. And note that, the time space Δt should be small enough. If Δt is not small enough, it may exceed the edge of the narrow band in one iteration step. In our experiment, $\Delta t = 0.0001$.

In Fig. (2), an ellipse is segmented in the artifical image. Figure (2a) is the image and the initial curve. Figure (2b) is segmented by the M-S model. Figure (2c) is segmented by the proposed model. In Fig. (3), the mirtral valve of the heart is segmented. Figure (3a) shows the ultrasound image of heart and the initil curve. Figure (3b) shows the segmentation result by M-S model. Figure (3c) shows the segmentation result by the proposed model. In Fig. (4), another heart image is segmented. From the following experiments, we can see that the two models have the similar segmentation results.

The comparison of computational time is given in Table 1. We can see that the two models spend the same iteration time to accomplish the segmentation process, but the proposed model consumes much less time than the M-S model in each iteration.

5 Conclusion

In the paper, we introduce a fast segmentation model called narrow band M-S model that combines the M-S model and narrow band scheme in a new initialization algorithm. The proposed model confines the calculation in a narrow hand region. So it is faster than M-S model, and achieves the same result as the old model.

Table 1 Time comparison between the M-S model and the proposed model

表 1 M-S 模型与新建模型的时间对比

Image	Method	Iteration times	Average time of one iteration
The artificial image	M-S model	50	0.911
	Proposed model	50	0.227
Heart image 1	M-S model	25	1.472
	Proposed model	25	0.371
Heart image 2	M-S model	100	0.963
	Proposed model	100	0.253

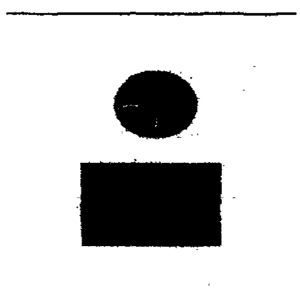

Fig. 2(a) An artificial image and the initial curve

图 2(a) 人工图像及初始曲线

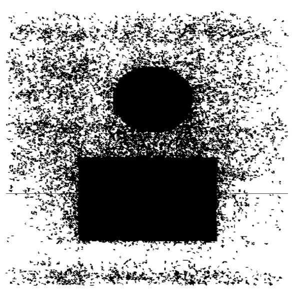

Fig. 2(b) Segmentation result by M-S model

图 2(b) M-S 的分割结果

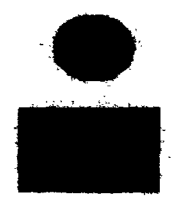

Fig. 2(c) Segmentation result by narrow band M-S model

图 2(c) 窄带 M-S 模型的分割结果

Fig. 3(a) Heart image 1 and the initial curve
图 3(a) 心脏图像及初始曲线

Fig. 3(b) Segmentation result by M-S model
图 3(b) M-S模型的分割结果

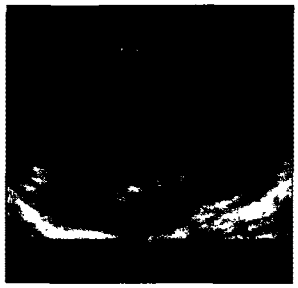

Fig. 3(c) Segmentation result by narrow band M-S model
图 3(c) 窄带 M-S模型的分割结果

Fig. 4(a) Heart image 2 and the initial curve
图 4(a) 心脏图像及初始曲线

Fig. 4(b) Segmentation result by M-S model
图 4(b) M-S模型的分割结果

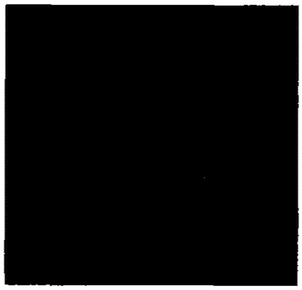

Fig. 4(c) Segmentation result by narrow band M-S model
图 4(c) 窄带 M-S模型的分割结果

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